ON BALANCING EVENT AND AREA COVERAGE
IN MOBILE SENSOR NETWORKS

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On Balancing Event and Area Coverage in Mobile Sensor Networks

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Abstract

In practice, the mobile sensor networks have two important tasks: firstly, sensors should be able to locate themselves close to where major events are happening so that event tracking becomes possible; secondly, the sensor networks should also maintain a good area coverage over the environment in order to detect new events. Because these two tasks are usually conflicting with each other, a coverage control policy should be able to balance the event and area coverage of the environment. However, most existing work is to achieve either optimal event coverage or optimal area coverage over the environment. In this thesis, a Voronoi-based coverage control with task assignment is introduced: each sensor is allowed to switch between event and area coverage depending on the intensity of events within its Voronoi cell, and both continuous-time and discrete-time control for sensor positions are discussed.

1 Introduction

In practice, the mobile sensor networks have two important tasks: firstly, when some major events are happening in the environment, sensors should be able to locate themselves close to those events so that event tracking becomes possible; Secondly, sensors should also maintain a good area coverage over the environment in order to detect new events. Because these two tasks are usually conflicting with each other: the networks do not have a good area coverage if most sensors are located close to certain events, a coverage control policy should be able to balance the event and area coverage. However, most existing work is to achieve either optimal event coverage or optimal area coverage over the environment, but not both at the same time.

The Voronoi-based approach consists an important part of event-driven coverage control for mobile sensor networks [1], and its extension to sensors with limited range [2], anisotropic [3] and directional [4,5] sensing footprint are discussed over the past years. Moreover, more practical problems such as collision avoidance in networks with disk-shape sensors [6] are drawing attentions. The optimal sensor deployment for such event-driven coverage control locates most sensors in the network near the event centers. When the events in the environment are concentrated at certain locations, then one would expect high density of sensors around such locations. On the other hand, to achieve optimal area coverage, sensors should be almost evenly distributed over the environment. For sensors with limited sensing range, the overlap between sensing footprints is minimized [7]. Such result is extended to sensors with directional sensing footprint [8], and area coverage in heterogeneous sensor networks [9] is also discussed. However, while both event and area coverage are extensively studied by many researchers, less attention is drawn to find an optimal sensor deployment that is favorable for both event tracking and area surveillance purposes.

To balance the event coverage and area coverage for the networks, it is natural to assign different task to each sensor, such that some sensors are specified to optimize event coverage around their neighborhood and others tend to maximize the area coverage. There is few work discussed Voronoi-based coverage control with different believes of event density among sensors [10,11], such a difference introduces additional computation on every Voronoi boundary, and makes it difficult to find a discrete-time update law for sensor positions.

1.1 Contribution of the Thesis

This thesis is inspired by the work on coverage control for Pan/Tilt/Zoom camera [5], on which I collaborated with Omur Arslan at Kod*Lab, University of Pennsylvania.

Because of the natural requirements for camera networks to spare sensing resources on events while maintaining the area coverage, Omur provided the research direction to extend the event coverage control for cameras to balancing the event and area coverage in the environment. However, it is difficult to find a straightforward task assignment scheme given the *conic Voronoi partition* of the workspace in [5], therefore we start from investigating task assignment for homogeneous mobile sensor networks with isotropic sensing footprint. I kept close collaboration with Omur when writing the review section for the event coverage control for mobile sensor networks: to proof the convergence of the discrete-time control for event coverage, Omur suggested to use techniques in MM-algorithm [13], which is also quite effective for proving useful results in other sections. The main contribution of the thesis is introducing task assignment scheme to the existing event coverage control, providing the continuous-time position control for sensors, and constructing a modified objective to allow discrete-time control, while existing work for similar problem settings [10, 11] only considered continuous-time control.

1.2 Organization of the Thesis

In this thesis, we at first review the case where sensors have the same task to optimize the event coverage of the environment, and provide both continuous-time and discrete-time control that are convergent to a locally optimal configuration. Then we extend the result to the case where sensors are assigned different tasks to either optimize event or area coverage, a continuous-time control following the gradient field of the objective function is proposed. Also, since iteratively updating sensor positions in this case is difficult, a discrete-time update law that optimizing an slightly modified objective is given. Lastly, the results for numerical simulation are given to show the effectiveness of task assignment in balancing event and area coverage.

2 Coverage Control Based on Gaussian Sensing Quality Model

2.1 Problem Statement

Throughout the thesis, we consider the homogeneous case where all sensors have the same sensing capability, and we assume that there is a shared belief of event distribution measuring the importance of each location in the environment among the sensors. Suppose the workspace $\mathcal{W}$ is a convex space in $\mathbb{R}^n$. A normalized event distribution $\phi_e(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$[1] is given, i.e. $\int_{\mathcal{W}} \phi_e(x)dx = 1$. Then we assume there are m mobile sensors labeled $i = 1, 2, \ldots, m$ with position $p_i \in \mathcal{W}$, and denote $\mathbf{p} := \{p_1, \cdots, p_m\}$ to be a *sensor configuration*. Given a specific position for sensor i, the sensor's sensing quality is defined as a measure on $\mathbb{R}^n$ with density:

$$q_i(x \mid p_i) = \exp\left(-\frac{\|x - p_i\|^2}{2\sigma^2}\right) \tag{2.1}$$

where σ is a fixed parameter characterizing the strength of sensing quality decaying along the radial direction.

Consider any partition $\mathcal{P} := \{P_1, P_2, \ldots, P_m\}$ of $\mathcal{W}$ [2], and each P_i is assigned to sensor i to be observed. Then the total sensing score [5] over the workspace can be defined as follow:

$$H_{\mathcal{P}} := \sum_{i=1}^{m} \int_{\mathcal{P}_i} \phi_e(x) q_i(x \mid p_i) dx \tag{2.2}$$

[1] $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x > 0\}$, is the positive reals

[2] a partition of $\mathcal{W}$ should satisfy that 1)$P_i \neq \emptyset, \forall i$; 2)$P_i^\circ \cap P_j^\circ, \forall i \neq j$, where P_i° is the interior of P_i; 3) $\bigcup_{i=1}^{m} P_i = \mathcal{W}$

observe that for a fixed sensor configuration $\mathbf{p}$, $H_{\mathcal{P}}$ is bounded above by:

$$H_{\mathcal{P}} \leq \int_{\mathcal{W}} \max_i \phi_e(x) q_i(x \mid p_i) dx \tag{2.3}$$

Based on this observation, we define the Voronoi partition $\mathcal{V} := \{V_1, V_2, \ldots, V_m\}$ of $\mathcal{W}$ as:

$$\begin{aligned} V_i(\mathbf{p}) : &= \{x \in \mathcal{W} \mid q_i(x \mid p_i) \geq q_j(x \mid p_j),\ \forall j \neq i\} \\ &= \{x \in \mathcal{W} \mid \|x - p_i\|^2 \leq \|x - p_j\|^2,\ \forall j \neq i\} \end{aligned} \tag{2.4}$$

It can be shown that V_i is convex has non-zero Lebesgue measure in $\mathbb{R}^n$ for all i if any two sensors are not located at same position [12]. In the rest of the thesis we only consider the proper configuration such that $p_i \neq p_j,\ \forall i \neq j$ to ensure the Voronoi partition of the workspace is well defined. The upper bound in (2.3) can be written as

$$\int_{\mathcal{W}} \max_i \phi_e(x) q_i(x \mid p_i) dx = \sum_{i=1}^{m} \int_{V_i} \phi_e(x) q_i(x \mid p_i) dx := H_{\mathcal{V}} \tag{2.5}$$

the equation above indicates that to maximize the overall sensing score $H_{\mathcal{P}}$ for any proper sensor configuration, the optimal partition of the workspace is the Voronoi partition $\mathcal{V}(\mathbf{p})$. As the each sensor update its position, the network should always keep the Voronoi partition of the workspace, therefore we have the following objective:

$$\max_{\mathbf{p} \in \mathcal{W}^m} \ H_{\mathcal{V}}(\mathbf{p}) = \sum_{i=1}^{m} \int_{V_i} \phi_e(x) q_i(x \mid p_i) dx \tag{2.6}$$

In Section 2.2, we provide a continuous-time control law that follows the gradient of $H_{\mathcal{V}}$ such that it converges to a locally optimal configuration. In Section 2.3, a MM(Minorization-Maximization) algorithm [13] based discrete-time "move-to-centroid" control is introduced, and it can be shown that $H_{\mathcal{V}}$ is non-decreasing after each step thus it eventually converges to a locally optimal sensor configuration.

2.2 Continuous-Time Coverage Control

To ensure that $H_{\mathcal{V}}(\mathbf{p})$ is monotonically increasing over time until it converges, a simple but effective continuous-time coverage control law is to let $\dot{p}_i$ follow the gradient of $H_{\mathcal{V}}(\mathbf{p})$ with respect to p_i. To compute the gradient of $H_{\mathcal{V}}(\mathbf{p})$, recall the Leibniz's rule:

Lemma 2.1 (Leibniz's Rule [14]). *Let $\Omega(p)$ be a region whose boundary depends on p smoothly, and the unit outward normal vector $n(p)$ is uniquely defined almost everywhere on the boundary $\partial\Omega$. Let $f(x,p)$ be a smooth function of p, and define*

$$I = \int_{\Omega(p)} f(x,p)dx$$

if we denote $\frac{\partial x}{\partial p}$ be the gradient of boundary point $x \in \partial\Omega$ with respect to p, then

$$\frac{\partial I}{\partial p} = \int_{\Omega(p)} \frac{\partial f(x,p)}{\partial p} dx + \int_{\partial\Omega(p)} f(x,p) n^{\mathrm{T}}(x) \frac{\partial x}{\partial p} dx$$

Firstly, for a given partition $\mathcal{P} = \{P_i, 1 \le i \le m\}$ of workspace $\mathcal{W}$, let its *mass* and *centroidal position* with respect to the event distribution defined as follow:

$$M_e(P_i, p_i) := \int_{P_i} \phi_e(x) \exp\left(-\frac{\|x - p_i\|^2}{2\sigma^2}\right) dx \tag{2.7a}$$

$$p_e(P_i, p_i) := \frac{1}{M_e(P_i, p_i)} \int_{P_i} \phi_e(x) \exp\left(-\frac{\|x - p_i\|^2}{2\sigma^2}\right) x dx \tag{2.7b}$$

notice that $M_e(P_i, p_i) > 0$ as long as P_i has non-zero Lebesgue measure in $\mathbb{R}^n$ because the event density and sensing quality is positive for any point in the workspace. Also, to simplify the expression, let $M_{e,i} := M_e(V_i(\mathbf{p}), p_i)$ and $p_{e,i} := p_e(V_i(\mathbf{p}), p_i)$. Once we have such centroidal notion, it can be shown that the gradient of the coverage objective function with respect to sensor i's position p_i is always directed to the centroidal position of its respective Voronoi cell:

Theorem 2.2. *Given the centroidal definition* (2.7)*, the gradient of* $H_{\mathcal{V}}(\mathbf{p})$ *with respect to sensor configuration variables are given by:*

$$\frac{\partial H_{\mathcal{V}}}{\partial p_i} = \frac{M_{e,i}}{\sigma^2}(p_{e,i} - p_i)^{\mathrm{T}} \tag{2.8}$$

Proof. Denote the outward normal vector of V_i at boundary point $x \in \partial V_i$ as $n_i(x)$. Notice that such normal vector is well defined except for some measure zero set where the Voronoi boundaries intersect with each other, then by Leibniz's rule, we have:

$$\begin{aligned}
\frac{\partial H_{\mathcal{V}}}{\partial p_i} &= \int_{V_i} \phi_e(x) \frac{\partial q_i(x \mid p_i)}{\partial p_i} dx + \int_{\partial V_i} \phi_e(x) q_i(x \mid p_i) n_i^{\mathrm{T}} \frac{\partial x}{\partial p_i} dx \\
&\quad + \sum_{j \neq i} \int_{\partial V_i \cap \partial V_j} \phi_e(x) q_j(x \mid p_j) n_j^{\mathrm{T}} \frac{\partial x}{\partial p_i} dx \\
&= \int_{V_i} \phi_e(x) \frac{\partial q_i(x \mid p_i)}{\partial p_i} dx \\
&\quad + \sum_{j \neq i} \int_{\partial V_i \cap \partial V_j} \left(\phi_e(x) q_i(x \mid p_i) n_i^{\mathrm{T}} \frac{\partial x}{\partial p_i} + \phi_e(x) q_j(x \mid p_j) n_j^{\mathrm{T}} \frac{\partial x}{\partial p_i} \right) dx \\
&\quad + \int_{\partial V_i \cap \partial \mathcal{W}} \phi_e(x) q_i(x \mid p_i) n_i^{\mathrm{T}} \frac{\partial x}{\partial p_i} dx
\end{aligned} \tag{2.9}$$

For any j such that $\partial V_i \cap \partial V_j \neq \emptyset$, by the definition of Voronoi cell (2.4), we have $q_i(x \mid p_i) = q_j(x \mid p_j)$ for points on the boundary $\partial V_i \cap \partial V_j$; on the other hand, since $n_i(x), n_j(x)$ are defined to be the outward normal vector of V_i, V_j respectively, we have $n_i(x) = -n_j(x)$. Therefore we conclude that:

$$\phi_e(x) q_i(x \mid p_i) n_i^{\mathrm{T}} \frac{\partial x}{\partial p_i} + \phi_e(x) q_j(x \mid p_j) n_j^{\mathrm{T}} \frac{\partial x}{\partial p_i} = 0, \ \forall x \in \partial V_i \cap \partial V_j \tag{2.10}$$

in addition, because the workspace boundary does not change as sensors move, we have:

$$n_i^{\mathrm{T}} \frac{\partial x}{\partial p_i} = 0, \ \forall x \in \partial V_i \cap \partial \mathcal{W} \tag{2.11}$$

substitute (2.10) and (2.11) into (2.9), the integrals on the Voronoi boundary $\partial V_i \cap \partial V_j$ for $\forall j \neq i$ and on the workspace boundary $\partial V_i \cap \partial \mathcal{W}$ vanishes , we get

$$\begin{aligned} \frac{\partial H_{\mathcal{V}}}{\partial p_i} &= \int_{V_i} \phi_e(x) \frac{\partial q_i(x \mid p_i)}{\partial p_i} dx \\ &= \int_{V_i} \phi_e(x) q_i(x \mid p_i) \frac{x - p_i}{\sigma^2} dx \\ &= \frac{M_{e,i}}{\sigma^2} (p_{e,i} - p_i)^{\mathrm{T}} \end{aligned} \tag{2.12a}$$

□

Theorem 2.2 shows that the gradient field of $H_{\mathcal{V}}(\mathbf{p})$ leads each sensor to move towards its respective Voronoi centroidal position, and when each sensor's position coincides with its Voronoi centroidal position, such sensor configuration is a critical point for $H_{\mathcal{V}}(\mathbf{p})$:

Corollary 2.3. *$H_{\mathcal{V}}(\mathbf{p})$ is locally optimal if and only if, the following holds:*

$$p_i = p_{e,i} \tag{2.13}$$

Once we compute the gradient of $\mathcal{H}_{\mathcal{V}}(\mathbf{p})$ with respect to each sensor position, the continuous-time "move-to-centroid" control law is given by:

$$\dot{p}_i = \frac{K_p}{\sigma^2}(p_{e,i} - p_i),\ 1 \le i \le m \tag{2.14}$$

where $K_p > 0$ is a fixed control parameter. The convergence under such control law is guaranteed by the following theorem:

Theorem 2.4. *Under the continuous-time "move-to-centroid" control law* (2.14)*, $H_{\mathcal{V}}(\mathbf{p})$ is non-decreasing and converges to a locally optimal point.*

Proof. when $\dot{p}_i$ is given by (2.14), the time-derivative of $H_{\mathcal{V}}(\mathbf{p})$ is:

$$\dot{H}_{\mathcal{V}}(\mathbf{p}) = K_p \sum_{i=1}^{m} M_{e,i} \left\|p_{e,i} - p_i\right\|^2 \ge 0$$

therefore $H_{\mathcal{V}}(\mathbf{p})$ is non-decreasing over time. Notice that $H_{\mathcal{V}}(\mathbf{p})$ is bounded above by some constant over $\mathbf{p} \in \mathcal{W}^m$, then by LaSalle's Invariant Principle [15], the sensor configuration $\mathbf{p}$ converges to some critical points where $\dot{H}_{\mathcal{V}}(\mathbf{p}) = 0$. □

2.3 Discrete-Time Coverage Control

In this section, we denote $\mathbf{p}[k] := \{p_1[k], \cdots, p_m[k]\}$ to be the current sensor configuration, $\mathbf{p}[k+1] := \{p_1[k+1], \cdots, p_m[k+1]\}$ to be the next sensor configuration under some discrete-time control law. Let $\mathcal{V}[k] := \mathcal{V}(\mathbf{p}[k]) = \{V_1[k], \cdots, V_m[k]\}$ be the Voronoi partition of $\mathcal{W}$ by current sensor configuration. Since the gradient field of $H_{\mathcal{V}}(\mathbf{p})$ always directs each sensor to its Voronoi centroidal position, intuitively one should apply the following discrete-time "move-to-centroid" control:

$$p_i[k+1] = p_i[k] + \lambda[p_e(V_i[k], p_i[k]) - p_i[k]] \tag{2.15}$$

where $0 < \lambda \leq 1$ is a fixed step size. By setting $\lambda = 1$, each sensor moves to its Voronoi centroidal position directly, while $\lambda < 1$ indicates a more conservative move towards the centroidal position. To show that under such discrete-time "move-to-centroid" control (2.15), $H_{\mathcal{V}}$ increases after each iteration i.e. $H_{\mathcal{V}[k+1]}(\mathbf{p}[k+1]) \geq H_{\mathcal{V}[k]}(\mathbf{p}[k])$, we firstly give the definition of *minorization functions*:

Definition 2.5. *Given a function $f(\theta) : \Theta \to \mathbb{R}$, a family of functions $g(\theta \mid \theta_m) : \Theta \to \mathbb{R}$ parameterized by θ_m is said to **minorize** $f(\theta)$ over Θ if $\forall \theta_m \in \Theta$:*

$$\begin{aligned} f(\theta) &\geq g(\theta \mid \theta_m), \ \forall \theta \in \Theta \\ f(\theta_m) &= g(\theta_m \mid \theta_m) \end{aligned} \tag{2.16}$$

*such $g(\theta \mid \theta_m)$ are the **minorization functions** of $f(\theta)$.*

The minorization functions are powerful tools to solve many optimization problems [13]: given $g(\theta \mid \theta_m)$ minorize $f(\theta)$ at θ_m, if one can find a non-decreasing step for $g(\theta \mid \theta_m)$, i.e. $\exists \overline{\theta}_m \ s.t. \ g(\overline{\theta}_m \mid \theta_m) \geq g(\theta_m \mid \theta_m)$, then $\theta_m \to \overline{\theta}_m$ is also a non-decreasing step for $f(\theta)$ by the inequalities: $f(\overline{\theta}_m) \geq g(\overline{\theta}_m \mid \theta_m) \geq g(\theta_m \mid \theta_m) = f(\theta_m)$. As the result, one can maximize $f(\theta)$ by maximizing $g(\theta \mid \theta_m)$ or finding a non-decreasing step for $g(\theta \mid \theta_m)$.

2.3.1 Minorization functions for $H_{\mathcal{V}}$

The discrete-time "move-to-centroid" control (2.15) implicitly defines the minorization function of $H_{\mathcal{V}}$ at current sensor configuration $\mathbf{p}[k]$:

$$H_k(\mathbf{p}) = \sum_{i=1}^{m} \int_{V_i[k]} \phi_e(x) q_i(x \mid p_i) dx \tag{2.17}$$

Lemma 2.6. *$H_k(\mathbf{p})$ minorize $H_{\mathcal{V}}(\mathbf{p})$ at $\mathbf{p}[k]$ for all $\mathbf{p} \in \mathcal{W}^m$*

Proof. For $\mathbf{p} \in \mathcal{W}^m$

$$\begin{aligned} H_k(\mathbf{p}) &= \int_{\mathcal{W}} \phi_e(x) \left[\sum_{i=1}^{m} q_i(x \mid p_i) \mathbf{1}_{x \in V_i[k]} \right] dx \\ &\leq \int_{\mathcal{W}} \phi_e(x) \max_{1 \leq i \leq m} q_i(x \mid p_i) dx \\ &= H_{\mathcal{V}}(\mathbf{p}) \end{aligned}$$

and it's obvious that $H_k(\mathbf{p}[k]) = H_{\mathcal{V}}(\mathbf{p}[k])$ □

Notice that $H_k(\mathbf{p})$ can be decoupled into optimization problems for each sensor i:

$$\max_{p_i \in \mathcal{W}} H_{ki}(p_i) = \int_{V_i[k]} \phi_e(x) q_i(x \mid p_i) dx$$

once we find $p_i[k+1]$ such that $H_{ki}(p_i[k+1]) \geq H_{ki}(p_i[k])$ for each sensor i, we can make sure $H_{\mathcal{V}}$ is non-deceasing. Now we only need to show that (2.15) is a non-decreasing step for $H_{ki}(p_i)$.

2.3.2 Non-decreasing step for H_{ki}

Firstly notice that $H_{ki}(p_i) = M_e(V_i[k], p_i)$. Since for sensor i such that $M_e(V_i[k], p_i[k]) = 0$, under (2.15) we have $p_i[k+1] = p_e(V_i[k], p_i[k]) = p_i[k]$, then $H_{ki}(p_i[k+1]) = H_{ki}(p_i[k]) = 0$ holds trivially. Therefore, we only consider the case where $M_e(V_i[k], p_i[k]) \neq 0$. Define a family of comparison functions as:

$$f_{ki}(p_i) = \log \left[\frac{M_e(V_i[k], p_i)}{M_e(V_i[k], p_i[k])} \right] = \log \left[\frac{\int_{V_i[k]} \phi_e(x) q_i(x \mid p_i) dx}{\int_{V_i[k]} \phi_e(x) q_i(x \mid p_i[k]) dx} \right] \tag{2.18}$$

$f_{ki}(p_i)$ are called comparison functions to $M_e(V_i[k], p_i)$ because we have the following:

Lemma 2.7. *Given $p_i[k]$ the current sensor configuration, and $p_i[k+1]$ the next* *sensor configuration, the following holds:*

$$M_e(V_i[k], p_i[k+1]) \geq M_e(V_i[k], p_i[k]) \text{ iff } f_{ki}(p_i[k+1]) \geq f_{ki}(p_i[k]) = 0$$

Proof. because the function $\log(x)$ is monotonically increasing. $\square$

Now we make sure $M_e(V_i[k], p_i)$ is non-decreasing by finding a non-decreasing step for $f_{ki}(p_i)$. Define the minorization functions for $f_{ki}(p_i)$ as:

$$g_{ki}(p_i) = \frac{1}{M_e(V_i[k], p_i[k])} \int_{V_i[k]} \phi_e(x) q_i(x \mid p_i[k]) \left(\frac{\|x - p_i[k]\|^2}{2\sigma^2} - \frac{\|x - p_i\|^2}{2\sigma^2} \right) dx \tag{2.19}$$

To be able to show that $g_{ki}(p_i)$ are minorization functions of $f_{ki}(p_i)$, we firstly recall the Jensen's inequality:

Lemma 2.8 (Jensen's Inequality)**.** *Suppose μ is a probability measure on $(\mathbb{R}^n, \mathcal{B})$, and $f : \mathbb{R}^n \to \mathbb{R}$ is μ-measurable. If φ is a convex function on $\mathbb{R}$, then following inequality holds:*

$$\varphi[\mathbb{E}_\mu(f(x))] \leq \mathbb{E}_\mu(\varphi \circ f(x)) \tag{2.20}$$

and if μ is with density function $p_\mu(x)$, the inequality can be written as:

$$\varphi\left[\int_{\mathbb{R}^n} f(x) p_\mu(x) dx\right] \leq \int_{\mathbb{R}^n} \varphi(f(x)) p_\mu(x) dx \tag{2.21}$$

Conversely, if φ is a concave function, the reverse of (2.20) *holds:*

$$\varphi[\mathbb{E}_\mu(f(x))] \geq \mathbb{E}_\mu(\varphi \circ f(x)) \tag{2.22}$$

then we show that $g_{ki}(p_i)$ have following properties:

Proposition 2.9. *$g_{ki}(p_i)$ minorize $f_{ki}(p_i)$ at $p_i[k]$.*

Proof. Observe that $\frac{\phi_e(x)q_i(x|p_i[k])}{M_e(V_i[k],p_i[k])}$ is a probability density function supported on $V_i[k]$. Since $\log(x)$ is a concave function, by Jensen's inequality:

$$\begin{aligned} f_{ki}(p_i) &= \log\left[\int_{V_i[k]} \frac{\phi_e(x)q_i(x \mid p_i[k])}{M_e(V_i[k],p_i[k])} \cdot \frac{q_i(x \mid p_i)}{q_i(x \mid p_i[k])} dx\right] \\ &\geq \int_{V_i[k]} \frac{\phi_e(x)q_i(x \mid p_i[k])}{M_e(V_i[k],p_i[k])} \cdot \log\left[\frac{q_i(x \mid p_i)}{q_i(x \mid p_i[k])}\right] dx \\ &= g_{ki}(p_i) \end{aligned} \tag{2.23}$$

also, $g_{ki}(p_i[k]) = f_{ki}(p_i[k]) = 0$, which finishes the proof. □

Proposition 2.10. *$g_{ki}(p_i)$ is a concave function of p_i, and centroidal definitions in* (2.7)*, we have*

$$p_e(V_i[k],p_i[k]) = \arg\max_{p_i} g_{ki}(p_i) \tag{2.24}$$

Proof. Let I_n be the $n \times n$ identity matrix, then the gradient and Hessian of $g_{ki}(p_i)$ with respect to p_i are given by:

$$\frac{\partial g_{ki}(p_i)}{\partial p_i} = \frac{1}{M_e(V_i[k],p_i[k])} \int_{V_i[k]} \phi_e(x)q_i(x \mid p_i[k]) \frac{x-p_i}{\sigma^2} dx \tag{2.25a}$$

$$\frac{\partial^2 g_{ki}(p_i)}{\partial p_i^2} = -\frac{I_n}{\sigma^2} \tag{2.25b}$$

Since $\frac{\partial^2 g_{ki}(p_i)}{\partial p_i^2}$ is negative definite, $g_{ki}(p_i)$ is a concave function of p_i; on the other hand, it's easy to check that $\left.\frac{\partial g_{ki}(p_i)}{\partial p_i}\right|_{p_e(V_i[k],p_i[k])} = 0$, then $p_e(V_i[k],p_i[k])$ is a global maxima of $g_{ki}(p_i)$. □

By proposition 2.9, we know that $g_{ki}(p_i)$ evaluated at any point on the line segment between $p_e(V_i[k],p_i[k])$ and $p_i[k]$ is greater than $g_{ki}(p_i[k])$ by its concavity:

Corollary 2.11. *Given $p_i[k+1] = p_i[k] + \lambda[p_e(V_i[k],p_i[k]) - p_i[k]]$, $0 < \lambda \leq 1$, we have:*

$$g_{ki}(p_i[k+1]) \geq g_{ki}(p_i[k])$$

Now we can proof that every time the sensor positions are updated according to (2.15), the objective value $H_{\mathcal{V}}$ is non-decreasing:

Theorem 2.12. *Under the discrete-time "move-to-centroid" control* (2.15)*, $H_{\mathcal{V}}$ is non-decreasing at each step.*

Proof. For all i, given the next position $p_i[k+1]$ for each sensor by (2.15), by Corollary 2.11, $g_{ki}(p_i[k+1]) \geq g_{ki}(p_i[k])$ holds. Because $g_{ki}(p_i)$ minorize $f_{ki}(p_i)$ at $p_i[k]$, we have $f_{ki}(p_i[k+1]) \geq g_{ki}(p_i[k+1]) \geq g_{ki}(p_i[k]) = f_{ki}(p_i[k])$, which indicates $M_e(V_i[k], p_i[k+1]) \geq M_e(V_i[k], p_i[k])$. Then $H_k(\mathbf{p}[k+1]) = \sum_{i=1}^m M_e(V_i[k], p_i[k+1]) \geq \sum_{i=1}^m M_e(V_i[k], p_i[k]) = H_k(\mathbf{p}[k])$. Lastly, because $H_k(\mathbf{p})$ minorize $H_{\mathcal{V}}(\mathbf{p})$ at $\mathbf{p}[k]$, we have $H_{\mathcal{V}}(\mathbf{p}[k+1]) \geq H_k(\mathbf{p}[k+1]) \geq H_k(\mathbf{p}[k]) = H_{\mathcal{V}}(\mathbf{p}[k])$ $\square$

Such discrete-time "move-to-centroid" control law can be applied in a distributed fashion: each sensor only needs to communicate with its Voronoi neighbors in order to compute its Voronoi centroidal position, thus it is effective for distributed mobile sensor networks.

3 Coverage Control with Task Assignment

Beside an event distribution with density $\phi_e(x)$ on the workspace, one can define an area distribution with density $\phi_a(x)$ as a uniform measure on $\mathcal{W}$, i.e. $\phi_a(x) \equiv \left(\int_{\mathcal{W}} dx\right)^{-1}$. Suppose now for each sensor, we define a task variable $s_i \in [0, 1]$, and its belief of activity distribution is defined to be a convex combination of the event distribution and the area distribution; the density of this activity distribution is:

$$\phi_i(x) = s_i \phi_e(x) + (1 - s_i)\phi_a(x) \tag{3.1}$$

under such construction, sensor i has a sensing score of $\phi_i(x)q_i(x \mid p_i)$ over the workspace, intuitively, the objective function to be maximized should be:

$$\hat{H}(\mathbf{p}, \mathbf{s}) = \int_{\mathcal{W}} \max_i [\phi_i(x)q_i(x \mid p_i)]dx \tag{3.2}$$

such objective function suggests that each point should assigned to the sensor that has the highest sensing score on that point. It requires each sensor to communicate with all the sensors in the network to achieve such assignment, which is impractical for most cases given the distributed nature of the mobile sensor networks. Therefore we should keep the Voronoi partition of the workspace. Notice that given the Voronoi cell definition in (2.4), $\hat{H}$ is lower bounded by:

$$\hat{H}(\mathbf{p}, \mathbf{s}) = \int_{\mathcal{W}} \max_i [\phi_i(x)q_i(x \mid p_i)]dx \geq \sum_{i=1}^{m} \int_{V_i} \phi_i(x)q_i(x \mid p_i)dx := \hat{H}_{\mathcal{V}} \tag{3.3}$$

thus we would like to have the following objective:

$$\max_{\substack{\mathbf{p} \in \mathcal{W}^m \\ \mathbf{s} \in [0,1]^m}} \quad \hat{H}_{\mathcal{V}}(\mathbf{p}, \mathbf{s}) = \sum_{i=1}^{m} \int_{V_i} [s_i\phi_e(x) + (1 - s_i)\phi_a(x)]q_i(x \mid p_i)dx \tag{3.4}$$

Similar to the case without task assignment, we provide a continuous-time control law that follows the gradient of $\hat{H}_{\mathcal{V}}$ such that it converges to a locally optimal configuration, and now the gradient is with additional terms due to the different activity distribution on each side of the Voronoi boundary. In Section 3.2, we argue that it is difficult to find a discrete-time control due to the additional terms on the boundary, and we provide a discrete control law for an alternative objective function instead.

3.1 Continuous-Time Coverage Control

We firstly take the gradient of $\hat{H}_{\mathcal{V}}$ with respect to p_i and,s_i. Since now the boundary terms cannot be canceled with each other, for each Voronoi boundary ∂V_i, we need to compute its outward normal vector $n_i(x)$ and the infinitesimal area changes $n_i^{\mathrm{T}}(x)\frac{\partial x}{\partial p_i}$ with respect to p_i :

Lemma 3.1. *Given sensor i, j s.t. $\partial V_i \cap \partial V_j \neq \emptyset$, for any point $x \in \partial V_i \cap \partial V_j$ where $n_i(x), n_j(x)$ is well defined, we have the following:*

$$n_i(x) = -n_j(x) = \frac{p_j - p_i}{\|p_j - p_i\|} \tag{3.5a}$$

$$n_i^{\mathrm{T}}(x)\frac{\partial x}{\partial p_i} = \frac{(x - p_i)^{\mathrm{T}}}{\|p_j - p_i\|} \tag{3.5b}$$

Proof. On $\partial V_i \cap \partial V_j$, we have:

$$\|x - p_i\|^2 - \|x - p_j\|^2 = 0 \tag{3.6}$$

To get the normal vector $n(x)$, suppose x is moving with rate $\dot{x}$, take time-derivative of (3.6), we have:

$$(p_j - p_i)^{\mathrm{T}}\dot{x} = 0$$

the equation above indicates that to stay at the boundary, $\dot{x}$ should be perpendicular to $(p_j - p_i)^{\mathrm{T}}$; on the other hand, $\dot{x}$ should be the tangent direction of the boundary. Then $\left(\frac{p_j - p_i}{\|p_j - p_i\|}\right)^{\mathrm{T}}$ is a normal vector of $\partial V_i \cap \partial V_j$; also, $\left(\frac{p_j - p_i}{\|p_j - p_i\|}\right)^{\mathrm{T}}$ is pointing outward V_i because it's a increasing direction for $\|x - p_i\|^2 - \|x - p_j\|^2$. Therefore one have (3.5a).

Now we take the gradient of (3.6) with respect to p_i:

$$(p_j - p_i)^{\mathrm{T}} \frac{\partial x}{\partial p_i} = (x - p_i)^{\mathrm{T}}$$

dividing both side by $\|p_j - p_i\|$, we get (3.5b). $\square$

Similarly, given any partition $\mathcal{P} = \{P_i, 1 \leq i \leq m\}$ and sensor configuration $\mathbf{p}$, we also define the centroidal notion for the area distribution:

$$M_a(P_i, p_i) := \int_{P_i} \phi_a(x) \exp\left(-\frac{\|x - p_i\|^2}{2\sigma^2}\right) dx \tag{3.7a}$$

$$p_a(P_i, p_i) := \frac{1}{M_a(P_i, p_i)} \int_{P_i} \phi_a(x) \exp\left(-\frac{\|x - p_i\|^2}{2\sigma^2}\right) x dx \tag{3.7b}$$

also, we let $M_{a,i} := M_a(V_i(\mathbf{p}), p_i)$ and $p_{a,i} := p_a(V_i(\mathbf{p}), p_i)$.

Additionally, for a Voronoi partition $\mathcal{V}$ and corresponding sensor configuration $\mathbf{p}$, we can define the *outward mass*, *inward mass*, *outward centroidal position* and *inward centroidal position* for each Voronoi boundary ∂V_i as:

$$\mu_i^+ := \sum_{j=1}^{m} \int_{\partial V_i \cap \partial V_j} \phi_i(x) \exp\left(-\frac{\|x - p_i\|^2}{2\sigma^2}\right) \frac{1}{\|p_j - p_i\|} dx \tag{3.8a}$$

$$\mu_i^- := \sum_{j=1}^{m} \int_{\partial V_i \cap \partial V_j} \phi_j(x) \exp\left(-\frac{\|x - p_i\|^2}{2\sigma^2}\right) \frac{1}{\|p_j - p_i\|} dx \tag{3.8b}$$

$$p_i^+ := \frac{1}{\mu_i^+} \sum_{j=1}^{m} \int_{\partial V_i \cap \partial V_j} \phi_i(x) \exp\left(-\frac{\|x - p_i\|^2}{2\sigma^2}\right) \frac{x}{\|p_j - p_i\|} dx \tag{3.8c}$$

$$p_i^- := \frac{1}{\mu_i^-} \sum_{j=1}^{m} \int_{\partial V_i \cap \partial V_j} \phi_j(x) \exp\left(-\frac{\|x - p_i\|^2}{2\sigma^2}\right) \frac{x}{\|p_j - p_i\|} dx \tag{3.8d}$$

Theorem 3.2. *Given the centroidal definitions* (2.7), (3.7) *and* (3.8), *the gradient* *of $\hat{H}_{\mathcal{V}}$ with respect to sensor configuration variables are given by:*

$$\frac{\partial \hat{H}_{\mathcal{V}}}{\partial p_i} = s_i \frac{M_{e,i}}{\sigma^2}(p_{e,i} - p_i)^{\mathrm{T}} + (1 - s_i)\frac{M_{a,i}}{\sigma^2}(p_{a,i} - p_i)^{\mathrm{T}}$$
$$+ \mu_i^{+}(p_i^{+} - p_i)^{\mathrm{T}} - \mu_i^{-}(p_i^{-} - p_i)^{\mathrm{T}} \tag{3.9a}$$
$$\frac{\partial \hat{H}_{\mathcal{V}}}{\partial s_i} = M_{e,i} - M_{a,i} \tag{3.9b}$$

Proof. When taking derivative of $\hat{H}_{\mathcal{V}}$ with respect to p_i, similar to the proof of Theorem 2.2, by Leibniz's rule, we have:

$$\begin{aligned}\frac{\partial \hat{H}_{\mathcal{V}}}{\partial p_i} &= s_i \int_{V_i} \phi_e(x) \frac{\partial q_i(x \mid p_i)}{\partial p_i} dx + (1 - s_i) \int_{V_i} \phi_a(x) \frac{\partial q_i(x \mid p_i)}{\partial p_i} dx \\ &+ \sum_{j \neq i} \int_{\partial V_i \cap \partial V_j} \left(\phi_i(x) q_i(x \mid p_i) n_i^{\mathrm{T}} \frac{\partial x}{\partial p_i} + \phi_j(x) q_j(x \mid p_j) n_j^{\mathrm{T}} \frac{\partial x}{\partial p_i} \right) dx \\ &+ \int_{\partial V_i \cap \partial \mathcal{W}} \phi_e(x) q_i(x \mid p_i) n_i^{\mathrm{T}} \frac{\partial x}{\partial p_i} dx\end{aligned} \tag{3.10}$$

by Lemma 3.1, for $x \in \partial V_i \cap \partial V_j, \forall j \neq i$, we have:

$$\phi_i(x) q_i(x \mid p_i) n_i^{\mathrm{T}} \frac{\partial x}{\partial p_i} + \phi_j(x) q_j(x \mid p_j) n_j^{\mathrm{T}} \frac{\partial x}{\partial p_i} = [\phi_i(x) - \phi_j(x)] q_i(x \mid p_i) \frac{(x - p_i)^{\mathrm{T}}}{\|p_j - p_i\|} \tag{3.11}$$

again, because the workspace boundary does not change as sensors move, we have:

$$n_i^{\mathrm{T}} \frac{\partial x}{\partial p_i} = 0, \ \forall x \in \partial V_i \cap \partial \mathcal{W} \tag{3.12}$$

substitute (3.11) and (3.12) into (3.10), rewrite it with centroidal definitions, then we get (3.9a).

When taking derivative of $\hat{H}_{\mathcal{V}}$ with respect to s_i, we get:

$$\frac{\partial \hat{H}_{\mathcal{V}}}{\partial s_i} = \int_{V_i} \phi_e(x) q_i(x \mid p_i) dx - \int_{V_i} \phi_a(x) q_i(x \mid p_i) dx = M_{e,i} - M_{a,i}$$

$\square$

Notice that in (3.9a), the gradient field of $\hat{H}_{\mathcal{V}}$ with respect to p_i is a weighted composition of four "forces" acting on p_i: moving towards centroidal position of event, centroidal position of area, outward centroidal position of the boundary, and moving away from inward centroidal position of the boundary. We can combine these "forces" with the following centroidal definitions:

$$\hat{M}_i = s_i \frac{M_{e,i}}{\sigma^2} + (1 - s_i)\frac{M_{a,i}}{\sigma^2} + \mu_i^+ + \mu_i^- \tag{3.13a}$$

$$\hat{p}_i = \frac{1}{\hat{M}_i}\left[s_i \frac{M_{e,i}}{\sigma^2} p_{e,i} + (1 - s_i)\frac{M_{a,i}}{\sigma^2} p_{a,i} + \mu_i^+ p_i^+ + \mu_i^-(2p_i - p_i^-)\right] \tag{3.13b}$$

then $\frac{\partial \hat{H}_{\mathcal{V}}}{\partial p_i}$ can be rewritten as:

$$\frac{\partial \hat{H}_{\mathcal{V}}}{\partial p_i} = \hat{M}_i(\hat{p}_i - p_i)^{\mathrm{T}} \tag{3.14}$$

Therefore the continuous-time control law can still act as moving p_i towards a combined centroidal position, we would like to have the following control input:

$$\dot{p}_i = K_p(\hat{p}_i - p_i)^{\mathrm{T}} \tag{3.15a}$$

$$\begin{aligned} \dot{s}_i &= K_s s_i (M_{e,i} - M_{a,i})\mathbf{1}_{M_{e,i} \le M_{a,i}} \\ &\quad + K_s(1 - s_i)(M_{e,i} - M_{a,i})\mathbf{1}_{M_{e,i} > M_{a,i}} \end{aligned} \tag{3.15b}$$

where $\mathbf{1}_A$ is defined to be an indicator function i.e. $\mathbf{1}_A = 1$ if A is true, and $\mathbf{1}_A = 0$ otherwise. Having such control law (3.15b) for s_i ensures that s_i takes value in $[0, 1]$. It is obvious that s_i will eventually be close to either 1 or 0, and its transient state does not have much effect on the final sensor configuration of the network. The convergence of such control system is guaranteed by:

Theorem 3.3. *Under the continuous-time control law* (3.15), $\hat{H}_{\mathcal{V}}(\mathbf{p},\mathbf{s})$ *is non-decreasing and converges to a locally optimal point.*

Proof. when $\dot{p}_i$, $\dot{s}_i$ is given by (3.15), the time-derivative of $\hat{H}_{\mathcal{V}}(\mathbf{p},\mathbf{s})$ is:

$$\frac{d}{dt}\hat{H}_{\mathcal{V}}(\mathbf{p},\mathbf{s}) = \sum_{i=1}^{m} K_p\hat{M}_i\|\hat{p}_i - p_i\|^2 + K_s(M_{e,i} - M_{a,i})^2[s_i\mathbf{1}_{M_{e,i}\le M_{a,i}} + (1-s_i)\mathbf{1}_{M_{e,i}>M_{a,i}}] \ge 0$$

therefore $\hat{H}_{\mathcal{V}}(\mathbf{p},\mathbf{s})$ is non-decreasing overtime. Again, $\hat{H}_{\mathcal{V}}(\mathbf{p},\mathbf{s})$ is bounded above by some constant over $\mathbf{p} \in \mathcal{W}^m$, $\mathbf{s} \in [0,1]^m$, then by LaSalle's Invariant Principle [15], the sensor configuration $\mathbf{p}$ and task assignment $\mathbf{s}$ converges to some critical points where $\frac{d}{dt}\hat{H}_{\mathcal{V}}(\mathbf{p},\mathbf{s}) = 0$. □

3.2 Discrete-Time Coverage Control with An Alternative Objective

It is hard to find a discrete-time control law for (3.4), particularly for the position p_i: one can easily see that finding minorization functions for $\hat{H}_{\mathcal{V}}(\mathbf{p},\mathbf{s})$ with respect to sensor configuration $\mathbf{p}$ by fixing current Voronoi partition, as in 2.3.1, can not be applied here. For each sensor to search for a non-decreasing step for $\hat{H}_{\mathcal{V}}$, it should be able to estimate the changes in activity distribution due to the Voronoi boundary changes, which requires a pre-knowledge of the movement of its neighbors, therefore maximizing $\hat{H}_{\mathcal{V}}$ can not be decoupled into sub-problems for each sensor. Thus we would like to introduce an slightly modified objective function that allows such decoupling.

Because the problem comes with the difficulties for each sensor to "predict" the changes on each Voronoi boundary solely by its movement, we'd like to force each sensor to have same activity distribution near the Voronoi boundary. Firstly let's define the neighboring set of each sensor:

Definition 3.4. *The **neighboring set** of sensor i is defined to be a ball centered at p_i with radii d_i: $U_i(p_i) := \{x \in \mathbb{R}^n \mid \|x - p_i\| < d_i\}$; Let their union to be $U(\mathbf{p}) := \bigcup_{i=1}^{m} U_i(p_i)$*

Definition 3.5. *A **conflict-free configuration** is defined to be the sensor configuration that is in the following set:*

$$\mathcal{F} := \{\mathbf{p} \in \mathcal{W}^m \mid U_i(p_i) \cap U_j(p_j) = \emptyset, \forall i \neq j\}$$

We assume sensors have neighboring set with same fixed radii i.e. $d_i = d$, $\forall i$, the following result can be extended to more general case where d_i is dynamically controlled. For each sensor, it sense its own belief of activity distribution ϕ_i only within its neighboring set, and the conflict-free configuration ensures that its neighboring set does not overlap with those of other sensors. Outside the neighboring sets, we'd like to let all sensors sense a shared activity distribution ϕ_m, with density function:

$$\phi_m(x) = \max\{\phi_a(x), \phi_e(x)\} \tag{3.16}$$

choosing this shared distribution as the point-wise maximum of event and area distribution is conservative because it has least risk of losing track of points with high density outside the neighborhood sets. Given such construction, we have an alternative objective function $\bar{H}_{\mathcal{V}}$:

$$\bar{H}_{\mathcal{V}}(\mathbf{p}, \mathbf{s}) := \sum_{i=1}^{m} \int_{V_i} [\mathbf{1}_{x \notin U_i(p_i)} \phi_m(x) + \mathbf{1}_{x \in U_i(p_i)} \phi_i(x)] q_i(x \mid p_i) dx \tag{3.17}$$

where $\phi_i(x)$ is still a convex combination of $\phi_e(x)$ and $\phi_a(x)$, as defined in (3.1). Once we constrain the sensor configuration within the conflict-free configuration set, the following holds:

Lemma 3.6. *Assume $d_i = d$, $\forall i$, then for a conflict-free configuration $\mathbf{p} \in \mathcal{F}$, $\bar{H}_{\mathcal{V}}$ can be rewritten as:*

$$\bar{H}_{\mathcal{V}}(\mathbf{p}, \mathbf{s}) = \int_{\mathcal{W}/U(\mathbf{p})} \phi_m(x) \max_i q_i(x \mid p_i) dx + \sum_{i=1}^{m} \int_{U_i(p_i)} \phi_i(x) q_i(x \mid p_i) dx \tag{3.18}$$

Proof. For all point $x \in U_i(p_i)$, we know that $x \in V_i(\mathbf{p})$, otherwise we would have $x \in U_j(p_j) \cap U_i(p_i)$ for some j, which contradicts that $\mathbf{p} \in \mathcal{F}$. Therefore we have $U_i(p_i) \subseteq V_i(\mathbf{p})$, $\forall i$. Also, notice that $U_i(p_i)$ are disjoint, then

$$\begin{aligned}\bar{H}_{\mathcal{V}}(\mathbf{p}, \mathbf{s}) &= \sum_{i=1}^{m} \int_{V_i(\mathbf{p})/U_i(p_i)} \phi_m(x) q_i(x \mid p_i) dx + \sum_{i=1}^{m} \int_{V_i(\mathbf{p})} \mathbf{1}_{x \in U_i(p_i)} \phi_i(x) q_i(x \mid p_i) dx \\ &= \int_{\mathcal{W}/U(\mathbf{p})} \phi_m(x) \max_i q_i(x \mid p_i) dx + \sum_{i=1}^{m} \int_{U_i(p_i)} \phi_i(x) q_i(x \mid p_i) dx\end{aligned}$$

□

based on such observation, our objective would be:

$$\max_{\substack{\mathbf{p} \in \mathcal{F} \\ \mathbf{s} \in [0,1]^m}} \bar{H}_{\mathcal{V}}(\mathbf{p}, \mathbf{s}) = \int_{\mathcal{W}/U(\mathbf{p})} \phi_m(x) \max_i q_i(x \mid p_i) dx + \sum_{i=1}^{m} \int_{U_i(p_i)} \phi_i(x) q_i(x \mid p_i) dx \tag{3.19}$$

Given current state $\mathbf{p}[k] \in \mathcal{F}$, $\mathbf{s}[k] \in [0, 1]$, and Voronoi partition $\mathcal{V}[k]$, we introduce discrete-time update laws for p_i, s_i that ensure $\bar{H}_{\mathcal{V}}$ is non-decreasing at each iteration.

3.2.1 Conditions on update laws for s_i

Assume sensor configuration $\mathbf{p}[k]$ fixed, any update law for task assignment $\mathbf{s}$ that satisfies the following conditions for all i will make sure $\bar{H}_{\mathcal{V}}$ is non-decreasing:

$$\begin{aligned} & s_i[k] \leq s_i[k+1] \leq 1, \quad if \;\, M_e(U_i(p_i[k]), p_i[k]) \geq M_a(U_i(p_i[k]), p_i[k]) \\ & 0 \leq s_i[k+1] \leq s_i[k], \quad \text{otherwise} \end{aligned} \tag{3.20}$$

Proposition 3.7. *Given $s_i[k+1]$ satisfying (3.20) for all $1 \leq i \leq m$, then:*

$$\bar{H}_{\mathcal{V}}(\mathbf{p}[k], \mathbf{s}[k+1]) \geq \bar{H}_{\mathcal{V}}(\mathbf{p}[k], \mathbf{s}[k])$$

Proof.

$$\begin{aligned} & \bar{H}_{\mathcal{V}}(\mathbf{p}[k], \mathbf{s}[k+1]) - \bar{H}_{\mathcal{V}}(\mathbf{p}[k], \mathbf{s}[k]) \\ =& \sum_{i=1}^{m} \int_{U_i(p_i[k])} (s_i[k+1] - s_i[k])[\phi_e(x) - \phi_a(x)] q_i(x \mid p_i[k]) dx \\ =& (s_i[k+1] - s_i[k])[M_e(U_i(p_i[k]), p_i[k]) - M_e(U_i(p_i[k]), p_i[k])] \geq 0 \end{aligned}$$

□

The intuition of such result is that the coverage objective becomes the maximization of a convex combination of $M_e(U_i(p_i[k]), p_i[k])$ and $M_e(U_i(p_i[k]), p_i[k])$ by controlling the weights, any $s_i[k+1]$ that gives more weight on the higher mass will increase the objective function.

3.2.2 Conditions on update laws for p_i

Assume task assignment $\mathbf{s}[k]$ fixed. Given current state $\mathbf{p}[k]$, $\mathcal{V}[k]$, we'd like to define the *conflict-free set* of each sensor:

Definition 3.8. *Given Voronoi partition $\mathcal{V}[k]$, the* ***conflict-free set*** *of sensor i is defined to be:*

$$\mathcal{F}_i[k] := \{p_i \in V_i[k] \mid U_i(p_i) \subseteq V_i[k]\} \tag{3.21}$$

Here we provide several useful results about this set:

Lemma 3.9. *Given the definition of conflict-free set of sensor i, if $\mathbf{p}[k] \in \mathcal{F}$, then we have the following properties:*

1. *$\mathcal{F}_i[k]$ in non-empty for all sensor i;*
2. *$\mathcal{F}_i[k]$ is a convex set for all sensor i;*
3. *$\mathcal{F}_1[k] \times \mathcal{F}_2[k] \times \cdots \times \mathcal{F}_m[k] \subseteq \mathcal{F}$.*

Proof.

1. if $\mathbf{p}[k] \in \mathcal{F}$, we know that for all i, $U_i(p_i[k]) \subseteq V_i[k]$, then $p_i[k] \in \mathcal{F}_i[k]$ for all i;
2. Firstly notice that $V_i[k]$ is a convex set because it is an intersection of half-spaces: $V_i[k] = \bigcup_j \{x \in \mathcal{W} \mid \|x - p_i\| \leq \|x - p_j\|\}$.
   Then we assume $p_{i1}, p_{i2} \in \mathcal{F}_i[k]$, for $p_{i3} = \lambda p_{i1} + (1-\lambda)p_{i2},\ \lambda \in [0,1]$, given any point $x \in U_i(p_{i3})$, one can verify that $x = \lambda(p_{i1} + x - p_{i3}) + (1-\lambda)(p_{i2} + x - p_{i3})$. Since $\|(p_{i1}+x-p_{i3})-p_{i1}\| = \|x-p_{i3}\| < d$, we have $(p_{i1}+x-p_{i3}) \in U_i(p_{i1}) \subseteq V_i[k]$ and similarly $(p_{i2} + x - p_{i3}) \in U_i(p_{i2}) \subseteq V_i[k]$. x is a convex combination of two points in $V_i[k]$ then we have $x \in V_i[k]$ by convexity. Then we conclude that $U_i(p_{i3}) \subseteq V_i[k]$, i.e. $p_{i3} \in \mathcal{F}_i[k]$, thus $\mathcal{F}_i[k]$ is a convex set;
3. Notice that if $U_i(p_i) \subseteq V_i[k]$ for some p_i, then $U_i(p_i) \subseteq V_i^\circ[k]$, where $V_i^\circ[k]$ is the interior of $V_i[k]$. To show that, if $\exists x \in U_i(p_i)$ such that $x \in \partial V_i[k]$, then because $U_i(p_i)$ is an open set, we have $U_\epsilon(x) = \{y \mid \|x - y\| < \epsilon\} \subseteq U_i(p_i)$ for some ϵ. the hyperplane $\partial V_i[k]$ would partition such open ball $U_\epsilon(x)$ into two open set,

one is strictly in $V_i[k]$ and another is outside $V_i[k]$, which leads to contradiction. given $U_i(p_i) \subseteq V_i^\circ[k]$ for all i, and the fact that $V_i^\circ[k]$ are disjoint sets, we have $U_i(p_i) \cap U_j(p_j) = \emptyset, \ \forall i \neq j$, therefore $\mathcal{F}_1[k] \times \mathcal{F}_2[k] \times \cdots \times \mathcal{F}_m[k] \subseteq \mathcal{F}$.

□

Now by fixing the current Vornoi partition $\mathcal{V}[k]$, we can define the minorization function for $\bar{H}_{\mathcal{V}}$ to be:

$$\bar{H}_k(\mathbf{p}) := \sum_{i=1}^{m} \int_{V_i[k]} [\mathbf{1}_{x \notin U_i(p_i)} \phi_m(x) + \mathbf{1}_{x \in U_i(p_i)} \phi_i(x)] q_i(x \mid p_i) dx \tag{3.22}$$

Lemma 3.10. *$\bar{H}_k(\mathbf{p}, \mathbf{s}[k])$ minorize $\bar{H}_{\mathcal{V}}(\mathbf{p}, \mathbf{s}[k])$ at $\mathbf{p}[k]$ for $\mathbf{p} \in \mathcal{F}_1[k] \times \mathcal{F}_2[k] \times \cdots \times \mathcal{F}_m[k]$*

Proof. Given $\mathbf{p} \in \mathcal{F}_1[k] \times \mathcal{F}_2[k] \times \cdots \times \mathcal{F}_m[k]$,

$$\begin{aligned}
\bar{H}_k(\mathbf{p}, \mathbf{s}[k]) &= \sum_{i=1}^{m} \int_{V_i[k]} [\mathbf{1}_{x \notin U_i(p_i)} \phi_m(x) + \mathbf{1}_{x \in U_i(p_i)} \phi_i(x)] q_i(x \mid p_i) dx \\
&= \int_{\mathcal{W}/U(\mathbf{p})} \phi_m(x) \left[\sum_{i=1}^{m} \mathbf{1}_{x \in V_i[k]} q_i(x \mid p_i) \right] dx + \sum_{i=1}^{m} \int_{U_i(p_i)} \phi_i(x) q_i(x \mid p_i) dx \\
&\leq \int_{\mathcal{W}/U(\mathbf{p})} \phi_m(x) \max_i q_i(x \mid p_i) dx + \sum_{i=1}^{m} \int_{U_i(p_i)} \phi_i(x) q_i(x \mid p_i) dx \\
&= \bar{H}_{\mathcal{V}}(\mathbf{p}, \mathbf{s}[k])
\end{aligned}$$

Also, its easy to check that $\bar{H}_k(\mathbf{p}[k], \mathbf{s}[k]) = \bar{H}_{\mathcal{V}}(\mathbf{p}[k], \mathbf{s}[k])$. □

Notice that maximizing $\bar{H}_k(\mathbf{p}, \mathbf{s}[k])$ within $\mathbf{p} \in \mathcal{F}_1[k] \times \mathcal{F}_2[k] \times \cdots \times \mathcal{F}_m[k]$ is equivalent to the following objective for each sensor i:

$$\max_{p_i \in \mathcal{F}_i[k]} \bar{H}_{ki}(p_i) = \int_{V_i[k]} [\mathbf{1}_{x \notin U_i(p_i)} \phi_m(x) + \mathbf{1}_{x \in U_i(p_i)} \phi_i(x)] q_i(x \mid p_i) dx \tag{3.23}$$

the update law for p_i can be any non-decreasing step for $\bar{H}_{ki}(p_i)$:

Proposition 3.11. *if* $\bar{H}_{ki}(p_i[k+1]) \geq \bar{H}_{ki}(p_i[k])$*, for all* $1 \leq i \leq m$*, then*

$$\bar{H}_{\mathcal{V}}(\mathbf{p}[k+1], \mathbf{s}[k]) \geq \bar{H}_{\mathcal{V}}(\mathbf{p}[k], \mathbf{s}[k])$$

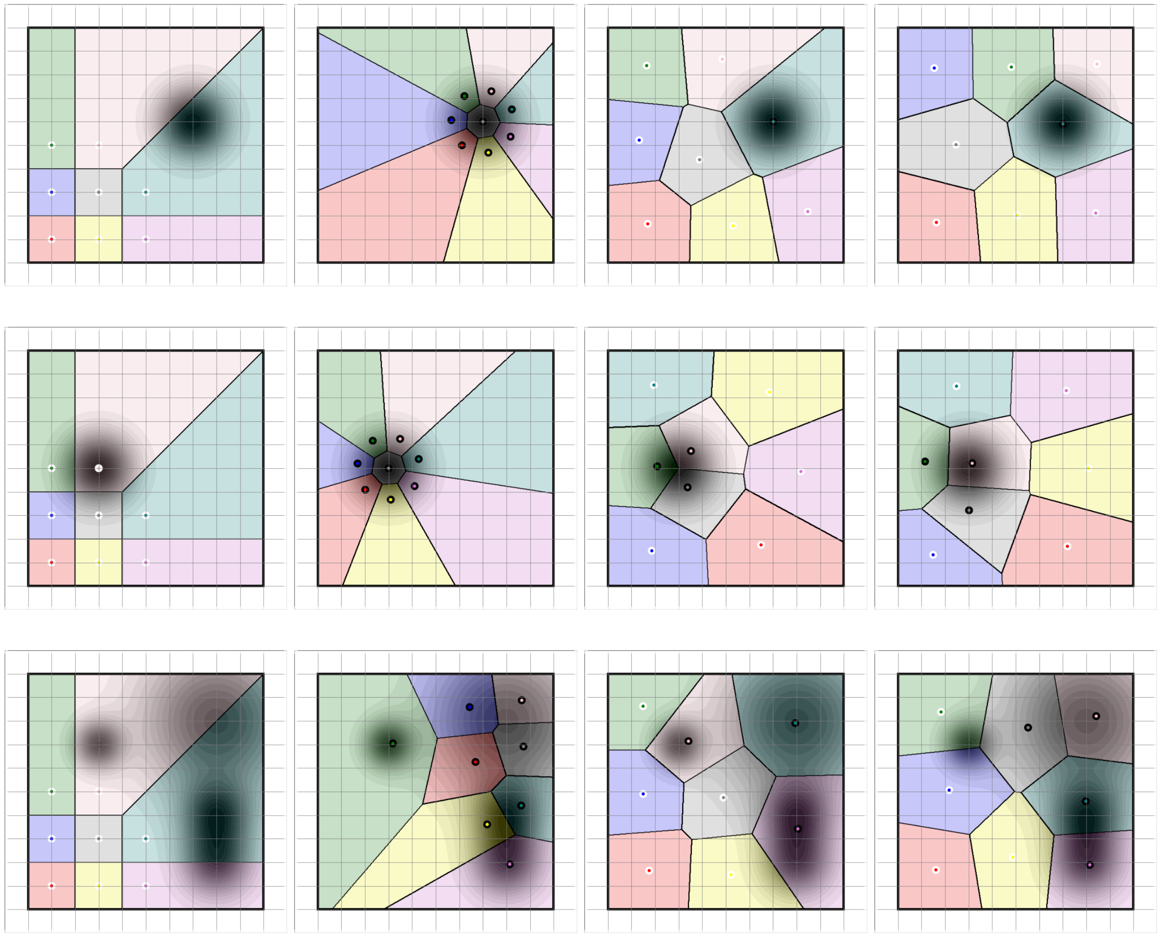
Proof. Given $\bar{H}_{ki}(p_i[k+1]) \geq \bar{H}_{ki}(p_i[k])$ for all i, we have $\bar{H}_k(\mathbf{p}[k+1], \mathbf{s}[k]) = \sum_{i=1}^{m} \bar{H}_{ki}(p_i[k+1]) \geq \sum_{i=1}^{m} \bar{H}_{ki}(p_i[k]) = \bar{H}_k(\mathbf{p}[k], \mathbf{s}[k])$. Because $\bar{H}_k(\mathbf{p}, \mathbf{s}[k])$ minorize $\bar{H}_{\mathcal{V}}(\mathbf{p}, \mathbf{s}[k])$ at $\mathbf{p}[k]$, we have $\bar{H}_{\mathcal{V}}(\mathbf{p}[k+1], \mathbf{s}[k]) \geq \bar{H}_k(\mathbf{p}[k+1], \mathbf{s}[k]) \geq \bar{H}_k(\mathbf{p}[k], \mathbf{s}[k]) = \bar{H}_{\mathcal{V}}(\mathbf{p}[k], \mathbf{s}[k])$. □

Unfortunately, we cannot find global maximum of (3.23) with the centroidal notions. Each sensor should do line search to find a non-decreasing step, we discuss one way to do the line search in Appendix.

4 Simulation Result

In this section, numerical simulation results are provided to demonstrate the locally optimal sensor configurations under different settings, from which we see significant improvement in balancing event and area coverage by introducing task assignment for each sensor.

In the simulation settings, the workspace is a 10×10 square region. the sensor network is composed of 8 sensors with identical Gaussian sensing quality model with parameter $\sigma = 1$. Under different event distributions, we compare the optimal sensor configuration for the following three proposed coverage control approaches: 1)discrete-time "move-to-centroid" control for event coverage only, with control parameter $\lambda = 1$, i.e. each sensor directly move to its Voronoi centroidal position at every iteration; 2)continuous-time control for coverage control with task assignment, with control pa-



(a) (b) (c) (d)

Figure 1: Visualization of locally optimal sensor configuration for different cases: (a)initial sensor configuration; (b)locally optimal sensor configuration under discrete-time "move-to-centroid" control without task assignment; (c)locally optimal sensor configuration under continuous-time "move-to-centroid" control with task assignment; (d)locally optimal sensor configuration under discrete-time control for the alternative objective with task assignment. Sensors assigned to event coverage are shown with black edges, and sensors assigned to area coverage are with white edges.

rameter $K_p = 3, K_s = 200$; 3)discrete-time control for coverage control for alternative objective with task assignment, with neighborhood set radii $d = 1$, each sensor take a non-decreasing step computed by line search as described in Appendix. Firstly, we consider the cases where a single event happens in the workspace. The first row of Figure.1 shows the case where sensors are initially located away from the event. The density function of the event is given by:

$$\phi_{e1}(x) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left\|x - [7,6]^{\mathrm{T}}\right\|^2\right)$$

The simulation result shows significant difference in optimal sensor configuration after task assignment is introduced: The optimal configuration for event coverage only locates all the sensors close to the event, and most Voronoi cells are extremely "imbalanced", i.e. the geometric centroid of the Voronoi cell is far away from the position of its corresponding sensor. Such configuration is unfavorable for area coverage because the points away from the event are sensed with low quality due to the imbalanced Voronoi cell. On the other hand, when the task assignment is introduced, most sensors are on area coverage and only one sensor is located near the event center to do event coverage. The sensor on area coverage is prevented from getting close to the event for two reason: one is that the major force acted on the sensor is directed towards the geometric center of its Voronoi cell instead of event center; another is that the sensor is pushed away from event by additional force on either the Voronoi boundary or the neighborhood set boundary due to the difference in activity distribution on two sides of the boundary. Based on such observation, one can expect that initial sensor configuration will have effect on how many sensors are on event coverage in the optimal configuration.

To show this, the second row of Figure.1 demonstrate the case where sensors are initially close to the event. The density function of the event is given by:

$$\phi_{e2}(x) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left\|x - [3,5]^{\mathrm{T}}\right\|^2\right)$$

Since now there are 3 sensors that is initially located near the event, those sensors immediately switch to event coverage and stay close to the event. Notice that under the discrete-time control for the slightly modified objective, the event is not evenly assigned to those 3 sensors, as in the continuous-time control case. This is because of the additional constraints that the neighborhood sets of sensors should not intersect. Lastly, we shows the optimal configuration for a more complex event function, as in the third row of Figure.1. The density function of the event is given by:

$$\begin{aligned}\phi_{e3}(x) = \frac{1}{C}\Bigg[&\exp\left(-\frac{1}{9}\left\|x-[8,8]^{\mathrm{T}}\right\|^2\right)+\exp\left(-\frac{1}{2}\left\|x-[8,2]^{\mathrm{T}}\right\|^2\right)\\ &+\exp\left(-\frac{1}{2}\left\|x-[8,4]^{\mathrm{T}}\right\|^2\right)+\exp\left(-\left\|x-[3,7]^{\mathrm{T}}\right\|^2\right)\Bigg]\end{aligned}$$

where C is a normalizing term. By introducing the task assignment, the continuous-time control achieves an optimal configuration such that each event intensive region is occupied by one sensor, since initially no sensor is close to events. However, the optimal configuration for discrete-time control with alternative objective is not ideal: for the event at $[3,7]^{\mathrm{T}}$, although there are sensors located around the event, no sensor is on event coverage and close to event center. The reason is because no sensor's neighborhood set is close enough to the event so that sensor can switch to event coverage. Such result shows for the coverage control with task assignment, having modified objective with additional constraints for computational efficiency comes with disadvantage of potentially getting less optimal configuration.

5 Conclusion

In this thesis, we proposed the Voronoi-based coverage control with task assignment to balance the event and area coverage in the workspace. A continuous-time "move-to-centroid" control is shown to converge to a locally optimal sensor configuration; Then a discrete-time control for a slightly modified objective is introduced. The simulation result shows significant improvement in balancing event and area coverage by introducing the task assignment to each sensor. It can be observed that in the optimal configuration, each sensor has distinct behavior that matches its task: the sensor assigned to event coverage is located near event center, while the sensor on area coverage positions itself close to the geometric center of its Voronoi cell.

For the discrete-time control with task assignment, potential drawbacks are seen in the simulation as the neighborhood sets of sensors introduce additional constraints for sensor movement. Dynamically controlling the radii of the neighborhood set for each sensor should be a possible improvement in the future work to reduce the effect of such constraints.

Another future research direction is to extend the idea of task assignment to camera networks [5]. As widely utilized in applications such as visual surveillance, the camera networks clearly need to effectively track event while maintain overall area coverage of the environment. A good task assignment scheme along with Pan/Tilt/Zoom control would allow only a few cameras in the network to fulfill complex sensing tasks, which is of great importance for many practical applications.

Appendix

A Non-decreasing step for $\bar{H}_{ki}(p_i)$ by line search

Recall In Section 3.2, the objective for sensor i is:

$$\max_{p_i \in \mathcal{F}_i[k]} \bar{H}_{ki}(p_i) = \int_{V_i[k]} [\mathbf{1}_{x \notin U_i(p_i)} \phi_m(x) + \mathbf{1}_{x \in U_i(p_i)} \phi_i(x)] q_i(x \mid p_i) dx \tag{A.1}$$

To do line search for $p_i[k+1]$ along the gradient direction, we firstly compute the gradient of $\bar{H}_{ki}(p_i)$ with respect to p_i:

Proposition A.1. *The gradient of $\bar{H}_{ki}(p_i)$ with respect to p_i is given by:*

$$\begin{aligned}\frac{\partial \bar{H}_{ki}(p_i)}{\partial p_i} &= \int_{V_i[k]} [\mathbf{1}_{x \notin U_i(p_i)} \phi_m(x) + \mathbf{1}_{x \in U_i(p_i)} \phi_i(x)] q_i(x \mid p_i) \frac{x - p_i}{\sigma^2} dx \\ &\quad + \int_{\partial U_i(p_i)} [\phi_i(x) - \phi_m(x)] q_i(x \mid p_i) \frac{x - p_i}{d} dx\end{aligned} \tag{A.2}$$

Proof. Divide $\bar{H}_{ki}(p_i)$ into two integrals:

$$\bar{H}_{ki}(p_i) = \int_{V_i[k]} \phi_m(x) q_i(x \mid p_i) dx + \int_{U_i(p_i)} [\phi_i - \phi_m(x)] q_i(x \mid p_i) dx$$

For the second term, similar to Lemma 3.1, we can easily verify that on $\partial U_i(p_i)$, $n^{\mathrm{T}}(x) = n^{\mathrm{T}}(x) \frac{\partial x}{\partial p_i} = \frac{x - p_i}{d}$. Then we take the gradient of two integrals respectively by Leibniz's Rule and we get (A.2). □

To rewrite the gradient as a vector direct from p_i to some centroidal position, we define the following masses, centroidal positions:

$$M_{ki} = \int_{V_i[k]} [\mathbf{1}_{x \notin U_i(p_i)} \phi_m(x) + \mathbf{1}_{x \in U_i(p_i)} \phi_i(x)] q_i(x \mid p_i) dx \tag{A.3a}$$

$$\mu_{ki}^{+} = \int_{\partial U_i(p_i)} \phi_i(x) q_i(x \mid p_i) dx, \quad \mu_{ki}^{-} = \int_{\partial U_i(p_i)} \phi_m(x) q_i(x \mid p_i) dx \tag{A.3b}$$

$$\bar{M}_{ki} = \frac{M_{ki}}{\sigma^2} + \frac{\mu_{ki}^{+} + \mu_{ki}^{-}}{d} \tag{A.3c}$$

$$p_{ki} = \int_{V_i[k]} [\mathbf{1}_{x \notin U_i(p_i)} \phi_m(x) + \mathbf{1}_{x \in U_i(p_i)} \phi_i(x)] q_i(x \mid p_i) x dx \tag{A.3d}$$

$$p_{ki}^{+} = \int_{\partial U_i(p_i)} \phi_i(x) q_i(x \mid p_i) x dx, \quad p_{ki}^{-} = \int_{\partial U_i(p_i)} \phi_m(x) q_i(x \mid p_i) x dx \tag{A.3e}$$

$$\bar{p}_{ki} = \frac{1}{\bar{M}_{ki}} \left[\frac{M_{ki}}{\sigma^2} p_{ki} + \frac{\mu_{ki}^{+}}{d} p_{ki}^{+} + \frac{\mu_{ki}^{-}}{d} (2p_i - p_{ki}^{-}) \right] \tag{A.3f}$$

Then $\frac{\partial \bar{H}_{ki}(p_i)}{\partial p_i}$ can be rewritten as:

$$\frac{\partial \bar{H}_{ki}(p_i)}{\partial p_i} = \bar{M}_{ki}(\bar{p}_{ki} - p_i) \tag{A.4}$$

Although $\bar{p}_{ki}$ is not necessarily a non-decreasing step for $\bar{H}_{ki}(p_i)$, it suggest that we can do binary search starting from $\bar{p}_{ki}$: update next step to be the α-th quantile on the line segment from $p_i[k]$ to current step and check the objective value at each step until we find a non-decreasing one, where $0 < \alpha < 1$ is a fixed parameter. However, given the constraint that $p_i[k+1]$ must be within $\mathcal{F}_i[k]$, we should project $\bar{p}_{ki}$ onto $\mathcal{F}_i[k]$ before doing the line search, first define:

$$\lambda^* = \max\{0 < \lambda \leq 1 \mid \lambda \bar{p}_{ki} + (1-\lambda) p_i[k] \in \mathcal{F}_i[k]\}$$

let $\bar{p}^*_{ki} = \lambda^* \bar{p}_{ki} + (1-\lambda^*) p_i[k]$. Since $\mathcal{F}_i[k]$ is convex, any point on the line segment between $\bar{p}^*_{ki}$ and $p_i[k]$ is in $\mathcal{F}_i[k]$, thus $\bar{p}^*_{ki}$ can be a starting point of the binary line search.

References

[1] J. Cortes, S. Martinez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Transactions on robotics and Automation*, vol. 20, no. 2, pp. 243–255, 2004.

[2] J. Cortes, S. Martinez, and F. Bullo, "Spatially-distributed coverage optimization and control with limited-range interactions," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 11, no. 4, pp. 691–719, 2005.

[3] A. Gusrialdi, S. Hirche, D. Asikin, T. Hatanaka, and M. Fujita, "Voronoi-based coverage control with anisotropic sensors and experimental case study," *Intelligent Service Robotics*, vol. 2, no. 4, p. 195, 2009.

[4] K. Laventall and J. Cortés, "Coverage control by multi-robot networks with limited-range anisotropic sensory," *International Journal of Control*, vol. 82, no. 6, pp. 1113–1121, 2009.

[5] O. Arslan, H. Min, and D. E. Koditschek, "Voronoi-based coverage control of pan/tilt/zoom camera networks," in *Robotics and Automation (ICRA), 2018 IEEE International Conference on*, IEEE, 2018.

[6] O. Arslan and D. E. Koditschek, "Voronoi-based coverage control of heterogeneous disk-shaped robots," in *Robotics and Automation (ICRA), 2016 IEEE International Conference on*, pp. 4259–4266, IEEE, 2016.

[7] J. Stergiopoulos and A. Tzes, "Voronoi-based coverage optimization for mobile networks with limited sensing range-a directional search approach," in *American Control Conference, 2009. ACC'09.*, pp. 2642–2647, IEEE, 2009.

[8] T.-W. Sung and C.-S. Yang, "Voronoi-based coverage improvement approach for wireless directional sensor networks," *Journal of Network and Computer Applications*, vol. 39, pp. 202–213, 2014.

[9] N. Bartolini, T. Calamoneri, T. F. La Porta, and S. Silvestri, "Autonomous deployment of heterogeneous mobile sensors," *IEEE Transactions on Mobile Computing*, vol. 10, no. 6, pp. 753–766, 2011.

[10] F. Abbasi, A. Mesbahi, and J. M. Velni, "Coverage control of moving sensor networks with multiple regions of interest," in *American Control Conference (ACC), 2017*, pp. 3587–3592, IEEE, 2017.

[11] M. Santos, Y. Diaz-Mercado, and M. Egerstedt, "Coverage control for multi-robot teams with heterogeneous sensing capabilities," *IEEE Robotics and Automation Letters*, 2018.

[12] Q. Du, V. Faber, and M. Gunzburger, "Centroidal voronoi tessellations: Applications and algorithms," *SIAM Review*, vol. 41, no. 4, pp. 637–676, 1999.

[13] D. R. Hunter and K. Lange, "A tutorial on mm algorithms," *The American Statistician*, vol. 58, no. 1, pp. 30–37, 2004.

[14] H. Flanders, "Differentiation under the integral sign," *The American Mathematical Monthly*, vol. 80, no. 6, pp. 615–627, 1973.

[15] H. K. Khalil, "Noninear systems," *Prentice-Hall, New Jersey*, vol. 2, no. 5, pp. 5–1, 1996.